Revisiting Traditional Numerical Methods with Deep Learning

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Outline

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Total Variation

Rudin-Osher-Fatemi (ROF)

$$\inf_{u \in BV(\Omega)} \lambda \int_{\Omega} |\nabla u| + \frac{1}{2} \|Au - u_0\|^2_{L_2(\Omega)}$$

A denotes some kind of noising convolution.

original noisy image

result

Other kind of Regularization

Chambolle and Lion (CL) :

$$\inf_{u_1,u_2} \int_\Omega \nu_1 |\nabla u_1| + \nu_2 \left|\nabla^2 u_2\right| \mathrm{d}x + \frac{1}{2} \|A(u_1+u_2) - u_0\|^2_{L_2(\Omega)}$$

$$\left|\nabla^2 u_2\right| := \sqrt{|\partial_{xx} u_2|^2 + |\partial_{yy} u_2|^2 + 2|\partial_{xy} u_2|^2}$$

Restore image by $u = u_1 + u_2$

Adversarial Regularizers in Inverse Problems[5]

Goal: a learned regularization term parametrized by Θ

$$\text{argmin}_u \|Au - u_0\|_2^2 + \lambda \Psi_\Theta(u)$$

Intuition:

- For true (clean) data x, hope $\Psi_\Theta(x)$ is small
- For noise data x to be processed, hope $\Psi_\Theta(x)$ is large

Optimizing Θ like WGAN:

$$\min_{\Theta} \{\mathbb{E}_{X\sim\mathbb{P}_r}[\Psi_\Theta(X)] - \mathbb{E}_{X\sim\mathbb{P}_n}[\Psi_\Theta(X)]\}$$

Inference:

$$x \leftarrow x - \epsilon\nabla_x \left[\|Ax - y\|_2^2 + \lambda\Psi_\Theta(x)\right]$$

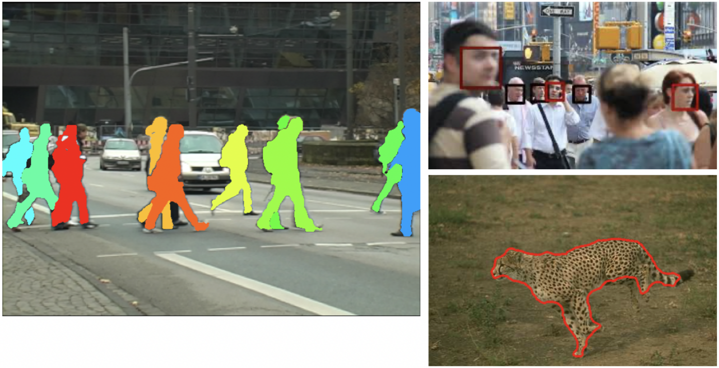
Outline

Image Restoration with Variation Regularization

Level Set Segmentation

Learning in Traditional Methods

Image Segmentation



Energy Methods

u_0 denotes the given image.

- Mumford-Shah:

$$\arg\min_{u,C} \mu \,\text{Length}(C) + \lambda \int_{\Omega} (u_0(x) - u(x))^2 dx + \int_{\Omega \setminus C} |\nabla u(x)|^2 dx$$

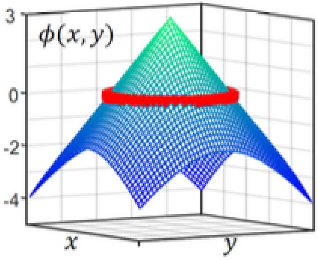
- Chan-Vese (CV) Model:

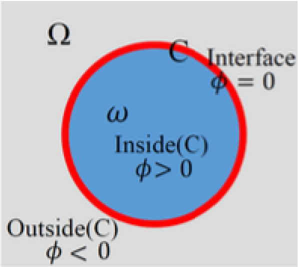
$$\arg\min_{c_1,c_2,C} \mu \text{ Length } (C) + \nu \text{ Area } (\text{inside}(C)) + \lambda_1 \int_{\text{inside } (C)} |u_0(x) - c_1|^2 \, dx + \lambda_2 \int_{\text{outside(C)}} |u_0(x) - c_2|^2 \, dx$$

Level Set

Denote the segmentation curve C as

$$C = \{x \in \Omega : \varphi(x) = 0\}$$





CV Model via Level Set Formulation

- CV Model:

$$\begin{aligned} \arg\min_{c_1, c_2, C} \; & \mu \text{ Length } (C) + \nu \text{ Area } (\text{inside}(C)) \\ & + \lambda_1 \int_{\text{inside } (C)} \left| u_0(x) - c_1 \right|^2 dx + \lambda_2 \int_{\text{outside(C)}} \left| u_0(x) - c_2 \right|^2 dx \end{aligned}$$

- Approximate CV energy functional via level set function φ as

$$\begin{aligned} \arg\min_{c_1, c_2, \varphi} \; & \mu \int_\Omega \delta(\varphi(x)) |\nabla \varphi(x)| dx + \nu \int_\Omega H(\varphi(x)) dx \\ & + \lambda_1 \int_\Omega \left| u_0(x) - c_1 \right|^2 H(\varphi(x)) dx + \lambda_2 \int_\Omega \left| u_0(x) - c_2 \right|^2 (1 - H(\varphi(x)) \end{aligned}$$

where $H(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0 \end{cases}$ and $\delta(t) = \frac{d}{dt} H(t)$

Solving CV Model

From the previous CV energy, we can obtain the gradient flow:

$$\frac{\partial \varphi}{\partial t} = \delta(\varphi)\left[\mu \operatorname{div}\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right) - \nu - \lambda_1 \left(u_0 - c_1(\varphi)\right)^2 + \lambda_2 \left(u_0 - c_2(\varphi)\right)^2\right]$$

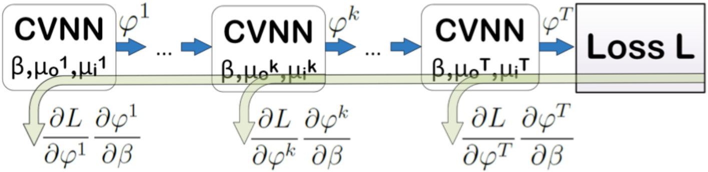
Optimization:

$$\varphi_{t+1} = \varphi_t + h \cdot \frac{\partial \varphi}{\partial t}$$

LEARNING CHAN-VESE [1]

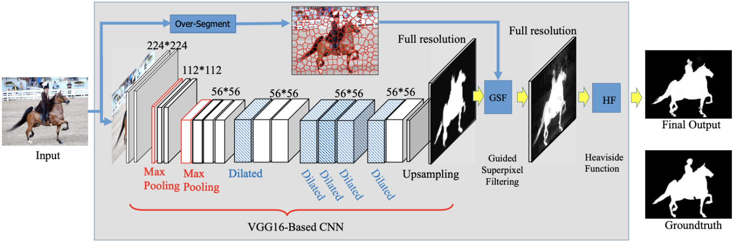
Replace the mean curvature term $\text{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)$ by a CNN $g(\varphi, \beta)$:

$$\varphi_{t+1} \leftarrow \varphi_t + h \cdot \left[\mu g(\varphi_t, \beta) - \nu - \lambda_1 (u_0 - c_1(\varphi_t))^2 + \lambda_2 (u_0 - c_2(\varphi_t))^2\right]$$



Deep Level Sets for Salient Object Detection[3]

Use a CNN to parametrize φ

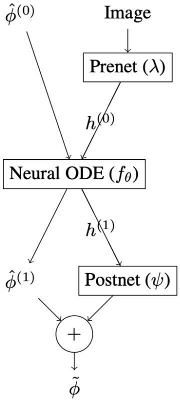


with CV energy as loss term.

Neural ODEs for Image Segmentation with Level Sets [2]

ϕ: level set function, h: image embedding

$$\gamma = (\hat{\phi}, h)$$
$$\frac{d\gamma}{dt} = f_\theta(\gamma, t) \text{ for } t \in [0,1]$$
$$\gamma^{(0)} = \left(\hat{\phi}^{(0)}, h^{(0)}\right)$$
$$\tilde{\phi} = \hat{\phi}^{(1)} + \psi\left(\gamma^{(1)}\right)$$



(a) Contour Evolution

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Learning to Discretize[6]

The key to solve conservation law

$$u_t(x,t) + f_x(u(x,t)) = 0$$

lies in the design of discretion scheme:

$$U_j^n - U_j^{n-1} = \Delta t \cdot -\frac{1}{\Delta x}\left(\hat{f}^n_{j+\frac{1}{2}} - \hat{f}^n_{j-\frac{1}{2}}\right),$$

where $U_j^n := u(x_j, t_n)$, $\hat{f}^n_{j-1} = \pi^f\left(U^{n-1}_{j-r-1}, U^{n-1}_{j-r}, \ldots, U^{n-1}_{j+s-1}\right)$, and π denotes specific scheme.

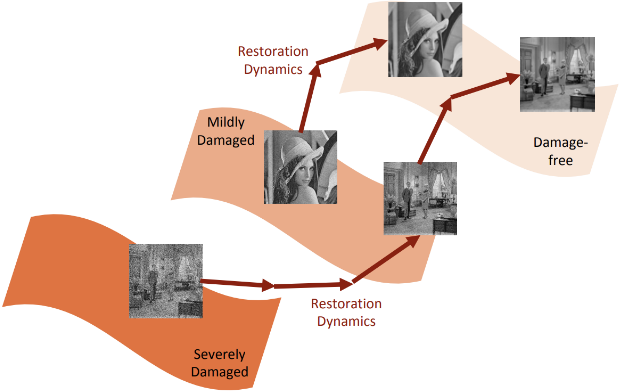
The idea is to use RL to learn a scheme.

- State: $\left(U_{j-r-1}^{n-1}, U_{j-r}^{n-1}, \ldots, U_{j+s-1}^{n-1}\right)$
- Action: the scheme
- Reward: distance between the RL approximation and ground truth

Reference: Learning to optimize [4] use RL to learn gradient descent scheme.

Dynamically Unfolding Recurrent Restorer (DURR) [7]

For image restoration with unknown degradation levels



DURR

use RL to decide whether to stop

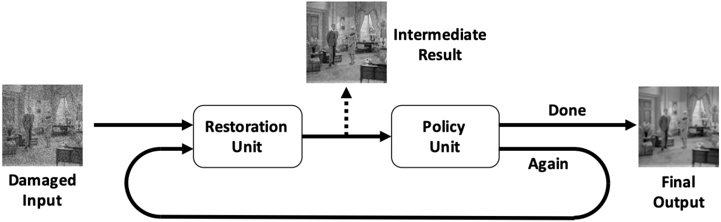


Figure 3: Pipeline of the dynamically unfolding recurrent restorer (DURR).

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